TTIC 31150/CMSC 31150
Mathematical Toolkit (Spring 2023)

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Lecture 5: The Real Spectral Theorem

Recap

- Eigenvectors and eigenvalues, eigenvectors of same eigenvalue form a subspace. Eigenvectors of different eigenvalues are linearly independent, inner products, norm, Cauchy-Schwartz.
- Gram-Schmidt orthogonalization, any finite-dimensional inner product space has an orthonormal basis.
- Properties of orthonormal bases: Fourier coefficients, Parseval's identity
- Adjoint of a linear transform
- Reisz representation theorem. Use to prove that every linear transformation has a unique adjoint
- Self-adjoint linear operators: eigenvalues are real, and eigenvectors corresponding to distinct eigenvalues are orthogonal.

The Real Spectral Theorem

Assume V is finite-dimensional

Theorem: every self-adjoint operator $\varphi: V \to V$ (which we know has real eigenvalues) has an orthonormal basis of eigenvectors (i.e., is "orthogonally diagonalizable").

- E.g., square symmetric matrices over $\mathbb{R}^n$.
- Gives a nice way to view action of such operators. Say φ has orthonormal eigenvectors $w_1, \ldots, w_n$ with associated eigenvalues $\lambda_1, \ldots, \lambda_n$. Then:

  For $v = \sum_i c_i w_i$, we have $\varphi(v) = \sum_i \lambda_i c_i w_i$.

  I.e., just stretching or shrinking in each "coordinate".

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Proof strategy:

1. Show that any such φ has at least one eigenvalue.

2. Use (1) to prove the theorem.

We'll do (2) first, then (1).

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Proof part 2: induction on dimension of V.

- Base-case: $\dim(V) = 1$. By part (1), there is at least one eigenvalue and eigenvector, so just scale the eigenvector to be unit-length.
- Let $\dim(V) = k + 1$. Let w be the eigenvector we are guaranteed by part (1) and let $W = span(\{w\})$. Let $W^{\perp} = \{v \in V: \langle v, w \rangle = 0\}$.
- Now, the idea to finish is to (a) show that $W^{\perp}$ is a subspace of V of dimension k, (b) show that φ restricted to $W^{\perp}$ is a self-adjoint operator on $W^{\perp}$ (and in particular maps $W^{\perp}$ to $W^{\perp}$), and (c) apply our inductive hypothesis to $W^{\perp}$ (which by design is orthogonal to w).

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(a): If $\langle v_1, w\rangle = 0$ and $\langle v_2, w\rangle = 0$ then $\langle a_1 v_1 + a_2 v_2, w\rangle = 0$, so it's a subspace. Dimension is k because a basis for $W^\perp \cup \{w\}$ is a basis for V.

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(b): If $\langle v, w \rangle = 0$ want to show that $\langle \varphi(v), w \rangle = 0$.

- We can use the fact that φ is self-adjoint and w is an eigenvector.
- $\langle \varphi(v), w \rangle = \langle v, \varphi(w) \rangle = \langle v, \lambda w \rangle = \lambda \langle v, w \rangle = 0$.

- Now, the idea to finish is to (a) show that $W^\perp$ is a subspace of V of dimension k, (b) show that φ restricted to $W^\perp$ is a self-adjoint operator on $W^\perp$ (and in particular maps $W^\perp$ to $W^\perp$), and (c) apply our inductive hypothesis to $W^\perp$ (which by design is orthogonal to w).

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(c): Now, just apply induction.

- Let $\{w_1, \dots, w_k\}$ be an orthonormal basis for $W^\perp$ of eigenvectors of φ restricted to $W^\perp$.
- So, $\left\{w_1, \dots, w_k, \frac{w}{\|w\|}\right\}$ is an orthonormal basis for V of eigenvectors of φ.

- Now, the idea to finish is to (a) show that $W^\perp$ is a subspace of V of dimension k, (b) show that φ restricted to $W^\perp$ is a self-adjoint operator on $W^\perp$ (and in particular maps $W^\perp$ to $W^\perp$), and (c) apply our inductive hypothesis to $W^\perp$ (which by design is orthogonal to w).

The Real Spectral Theorem

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Theorem: every self-adjoint operator $\varphi: V \to V$ (which we know has real eigenvalues) has an orthonormal basis of eigenvectors (i.e., is "orthogonally diagonalizable").

Proof strategy:

1. Show that any such φ has at least one eigenvalue.
2. Use (1) to prove the theorem.

Now, need to do (1).

Existence of eigenvalues

Let’s begin by assuming V is over $\mathbb{C}$. Then won’t need self-adjointness.

Example: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Existence of eigenvalues

Let's begin by assuming V is over $\mathbb{C}$. Then won't need self-adjointness.

Proposition 2.1 *Let V be a finite dimensional inner product space over $\mathbb{C}$ and let $\varphi : V \to V$ be a linear operator. Then φ has at least one eigenvalue.*

Proof: Let $\dim(V) = n$. Let $v \in V \setminus 0_V$ be any non-zero vector. Consider the set of $n+1$ vectors $\{v, \varphi(v), \varphi^2(v), \ldots, \varphi^n(v)\}$ where $\varphi^i(v) = \varphi(\varphi^{i-1}(v))$. Since the dimension of V is n, there must exist $c_0, \ldots, c_n \in \mathbb{C}$ not all 0 such that

$$c_0 \cdot v + c_1 \cdot \varphi(v) + \cdots + c_n \varphi^n(v) \;=\; 0_V \,.$$

For convenience, assume that $c_n \neq 0$, otherwise we can instead consider the sum to the largest i such that $c_i \neq 0$. What we want to do now is to factor the expression above into a product of degree-1 terms. This is where working over $\mathbb{C}$ will be useful.

Existence of eigenvalues

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OK, so we have $c_0 v + c_1\varphi(v) + \cdots + c_n\varphi^n(v) = 0_V$ with $c_n \neq 0$.

Let $P(x)$ denote the polynomial $c_0 + c_1x + \cdots + c_nx^n$. Then the above can be written as $(P(\varphi))(v) = 0$, where $P(\varphi) : V \to V$ is a linear operator defined as

$$P(\varphi) \;:=\; c_0 \cdot \mathrm{id} + c_1 \cdot \varphi + \cdots + c_n\varphi^n ,$$

with id used to denote the identity operator. Since P is a degree-n polynomial over $\mathbb{C}$, it can be factored into n linear factors, and we can write $P(x) = c_n \prod_{i=1}^{n}(x - \lambda_i)$ for $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. This means that we can write

$$P(\varphi) \;=\; c_n(\varphi - \lambda_n \cdot \mathrm{id}) \cdots (\varphi - \lambda_1 \cdot \mathrm{id}) .$$

Existence of eigenvalues

Let's begin by assuming V is over $\mathbb{C}$. Then won't need self-adjointness.

Proposition 2.1 *Let V be a finite dimensional inner product space over $\mathbb{C}$ and let $\varphi : V \to V$ be a linear operator. Then φ has at least one eigenvalue.*

OK, so we have $P(\varphi) = c_n(\varphi - \lambda_n \cdot id) \dots (\varphi - \lambda_1 \cdot id)$, and $P(\varphi)(v) = 0$.

Let $w_0 = v$ and define $w_i = \varphi(w_{i-1}) - \lambda_i \cdot w_{i-1}$ for $i \in [n]$. That is, we are working through the computation of $P(\varphi)(v)$ from right to left. Note that $w_0 = v \neq 0_V$ and $w_n = P(\varphi)(v) = 0_V$. Let i^* denote the largest index i such that $w_i \neq 0_V$. Then, we have

$$0_V = w_{i^*+1} = \varphi(w_{i^*}) - \lambda_{i^*+1} \cdot w_{i^*} .$$

This means that w_{i^*} is an eigenvector of φ with eigenvalue λ_{i^*+1}.

Existence of eigenvalues

Now, what about when V is over $\mathbb{R}$?

- Can do the same argument, except P now factors into linear and quadratic terms.
- Just need to show that we hit 0 in one of the linear terms, and not one of the irreducible quadratic terms.
- Specifically, want to show we don't get an equation of the form:

$$0_V = \varphi^2(w_{i^*}) + b\varphi(w_{i^*}) + cw_{i^*}, with\ b^2 < 4c$$

This is where self-adjointness comes in.

Existence of eigenvalues

Now, what about when V is over $\mathbb{R}$?

- Want to show we don't get an equation of the form:

$$0_V = \varphi^2(w_{i^*}) + b\varphi(w_{i^*}) + cw_{i^*}, with\ b^2 < 4c$$

$$\begin{aligned}
\langle w_{i^*}, \varphi^2(w_{i^*}) + b\varphi(w_{i^*}) + cw_{i^*}\rangle &= \langle w_{i^*}, \varphi^2(w_{i^*})\rangle + b\langle w_{i^*}, \varphi(w_{i^*})\rangle + c\langle w_{i^*}, w_{i^*}\rangle \\
&= \langle \varphi(w_{i^*}), \varphi(w_{i^*})\rangle + b\langle w_{i^*}, \varphi(w_{i^*})\rangle + c\langle w_{i^*}, w_{i^*}\rangle \\
&= \|\varphi(w_{i^*})\|^2 + b\langle w_{i^*}, \varphi(w_{i^*})\rangle + c\,\|w_{i^*}\|^2 \\
&\geq \|\varphi(w_{i^*})\|^2 - |b|\,\|w_{i^*}\|\,\|\varphi(w_{i^*})\| + c\,\|w_{i^*}\|^2 \\
&= \left(\|\varphi(w_{i^*})\| - \frac{|b|\,\|w_{i^*}\|}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)\|w_{i^*}\|^2 \\
&> 0.
\end{aligned}$$

So, the quadratic term can't be 0.

Raleigh Quotients

Definition 3.1 *Let $\varphi : V \to V$ be a self-adjoint linear operator and $v \in V \setminus \{0_V\}$. The* Rayleigh quotient *of φ at v is defined as*

$$\mathcal{R}_\varphi(v) \quad := \quad \frac{\langle v, \varphi(v) \rangle}{\|v\|^2} \,.$$

We can equivalently write $\mathcal{R}_\varphi(v) = \langle \hat{v}, \varphi(\hat{v}) \rangle$ for $\hat{v} = v / \|v\|$.

In other words, it is the length of the projection of $\varphi(\hat{v})$ onto $\hat{v}$.

If v was an eigenvector, then this would be the eigenvalue.

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Proposition 3.2 *Let $\dim(V) = n$ and let $\varphi : V \to V$ be a self-adjoint operator with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then,*

$$\lambda_1 \; = \; \max_{v \in V \setminus \{0_V\}} \mathcal{R}_\varphi(v) \quad \textit{and} \quad \lambda_n \; = \; \min_{v \in V \setminus \{0_V\}} \mathcal{R}_\varphi(v)$$

So, the vector v such that applying φ gives the largest "stretch" in the $\hat{v}$ direction is the eigenvector of largest eigenvalue, and likewise for the eigenvector of smallest eigenvalue.

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Proof: Let $w_1, \ldots, w_n$ be an orthonormal basis of evectors with evalues $\lambda_1, \ldots, \lambda_n$. Let $\hat{v} = \sum_i c_i w_i$. Then $\langle \hat{v}, \varphi(\hat{v}) \rangle = \langle \sum_i c_i w_i, \sum_i \lambda_i c_i w_i \rangle = \sum_i \lambda_i |c_i|^2$. Since $\sum_i |c_i|^2 = 1$, this is a weighted average of the λ_i's, and so is maximized at $c_1 = 1$, and minimized at $c_n = 1$.

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We can equivalently write $\mathcal{R}_\varphi(v) = \langle \hat{v}, \varphi(\hat{v}) \rangle$ for $\hat{v} = v / \|v\|$.

Extension / Generalization:

Proposition 3.3 (Courant-Fischer theorem) *Let $\dim(V) = n$ and let $\varphi : V \to V$ be a self-adjoint operator with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then,*

$$\lambda_k \;=\; \max_{\substack{S \subseteq V \\ \dim(S)=k}} \; \min_{v \in S \setminus \{0_V\}} \mathcal{R}_\varphi(v) \;=\; \min_{\substack{S \subseteq V \\ \dim(S)=n-k+1}} \; \max_{v \in S \setminus \{0_V\}} \mathcal{R}_\varphi(v) \, .$$

Positive Semidefiniteness

Definition 3.4 *Let $\varphi : V \to V$ be a self-adjoint operator. φ is said to be* positive semidefinite *if $\mathcal{R}_\varphi(v) \geq 0$ for all $v \neq 0$. φ is said to be* positive definite *if $\mathcal{R}_\varphi(v) > 0$ for all $v \neq 0$.*

Proposition 3.5 *Let $\varphi : V \to V$ be a self-adjoint linear operator. Then the following are equivalent:*

1. *$\mathcal{R}_\varphi(v) \geq 0$ for all $v \neq 0$.*
2. *All eigenvalues of φ are non-negative.*
3. *There exists $\alpha : V \to V$ such that $\varphi = \alpha^*\alpha$.*

Part of argument: if $\varphi = \alpha^*\alpha$ then $\langle v, \varphi(v)\rangle = \langle v, \alpha^*(\alpha(v))\rangle = \langle \alpha(v), \alpha(v)\rangle \geq 0$. This also means that if v is an eigenvector, its eigenvalue must be non-negative.

The decomposition of a positive semidefinite operator in the form $\varphi = \alpha^*\alpha$ is known as the Cholesky decomposition of the operator. Note that if we can write φ as $\alpha^*\alpha$ for any $\alpha : V \to W$, then this in fact also shows that φ is self-adjoint and positive semidefinite.